Global Max/Min

Consider a surface z = f(x,y) over a particular region R on the xy-plane.

An **absolute/global maximum** over R is the largest z-value over R.
An **absolute/global minimum** over R is the smallest z-value over R.

Key fact (Extreme value theorem)
The absolute max/min occur at either

1. A critical point, or
2. A boundary point.

Example: Let R be the triangular region in the xy-plane with corners at (0,-1), (0,1), and (2,-1). Above this triangular region, find the absolute max and min of

$$f(x,y) = \frac{1}{4}x + \frac{1}{2}y^2 - xy + 1$$

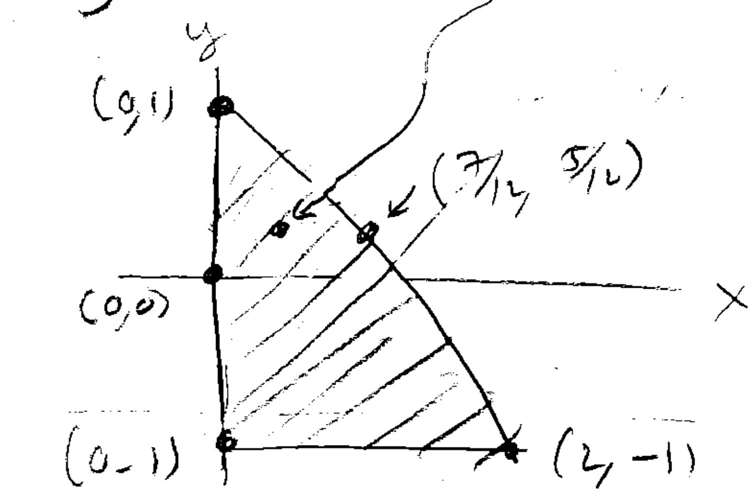
Entry Task

Do Step 1: Find the critical points

$f_x = \frac{1}{4} - y \overset{?}{=} 0 \Rightarrow y = 1/4$

$f_y = y - x \overset{?}{=} 0 \Rightarrow y = x$

$(x,y) = (1/4, 1/4)$



How to find the absolute max/min

Step 1: Find critical points inside region.

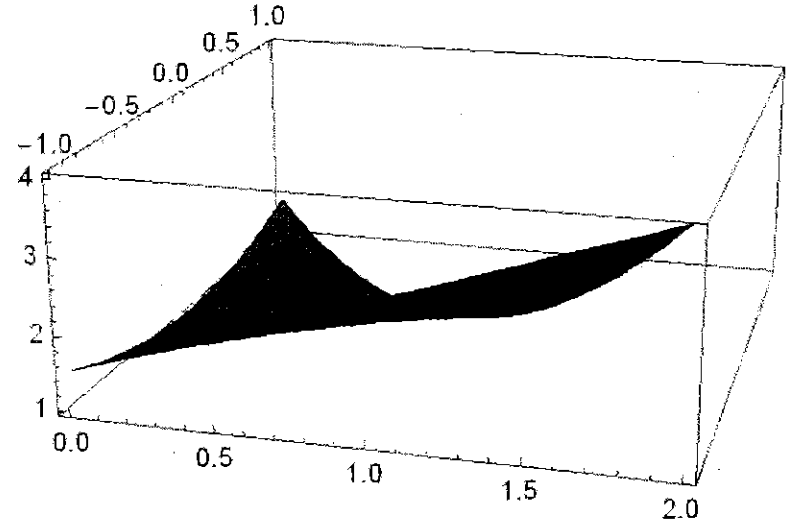
Step 2: Find critical numbers and corners above each boundary.

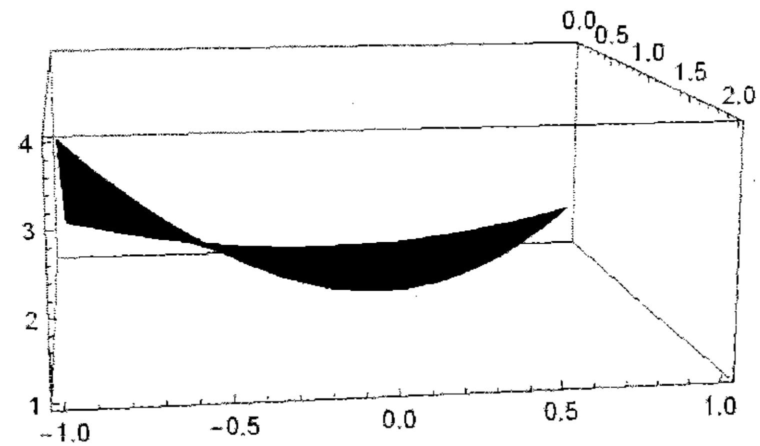
i) For each boundary, give an equation in terms of x and y. Find intersection with surface.

ii) Find critical numbers and endpoints for this one variable function. Label "corners".

Step 3: Evaluate the function at all points you found in steps 1 and 2.

Biggest output = global max
Smallest output = global min





$\boxed{A}$ $\boxed{x=0}$

$z = f(0,y) = \frac{1}{2}y^2 + 1, \quad -1 \le y \le 1 \Big\}$ A ONE VARIABLE ABS. MAX/MIN QUESTION!

$z' = y \overset{?}{=} 0 \Rightarrow y = 0$

NEED TO CONSIDER $y=0$, $\overbrace{y=-1, \text{ and } y=1}^{\text{ENDPOINTS ("CORNERS")}}$ ON THIS BOUNDARY.

$\boxed{B}$ $\boxed{y=-1}$

$z = f(x,-1) = \frac{1}{4}x + \frac{1}{2} + x + 1 \qquad 0 \le x \le 2$

$z = \frac{5}{4}x + 3/2$

$z' = 5/4 \overset{?}{=} 0 \leftarrow$ NEVER $\Rightarrow$ ONLY NEED TO CONSIDER ENDPOINTS ON THIS BOUNDARY

NEED TO CONSIDER $x=0$ AND $x=2$

$\boxed{C}$ LINE THRU $(0,1)$ AND $(2,-1) \Rightarrow m = \frac{-1-1}{2-0} = -1$

$\boxed{y=-x+1}$

$z = f(x,-x+1) = \frac{1}{4}x + \frac{1}{2}(-x+1)^2 - x(-x+1) + 1 = \frac{1}{4}x + \frac{1}{2}(-x+1)^2 + x^2 - x + 1$

$+ x^2 + 1 \qquad 0 \le x \le 2$

$z = -\frac{3}{4}x + \frac{1}{2}(-x+1)^2 + x^2 + 1$

$z' = -\frac{3}{4} - (-x+1) + 2x \overset{?}{=} 0 \Rightarrow -\frac{7}{4} + 3x = 0 \Rightarrow x = 7/12$

$x = 7/12 \Rightarrow y = -7/12 + 1 = 5/12$

NEED TO CONSIDER $(7/12, 5/12)$ ON THIS SIDE

(AND ENDPOINTS).

$\boxed{\text{CONCLUSION}}$

$z = f(0,0) = 1$

$z = f(0,1) = 3/2$

$z = f(0,-1) = 3/2$

$z = f(2,-1) = 4 \quad \left(\frac{1}{2} + \frac{1}{2} + 2 + 1\right)$ ABS. MAX

$z = f\left(\frac{1}{4}, \frac{1}{4}\right) = 1.03125$

$z = f\left(\frac{7}{12}, \frac{5}{12}\right) = 0.98958\overline{3}$ ABS. MIN

$z = f(0,0) = 1$

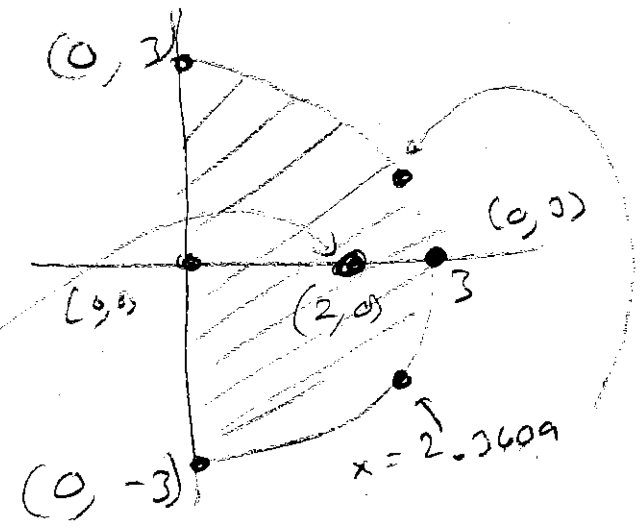
Example:

Find the absolute max/min of

$$f(x,y) = x^3 - 12x + y^2$$

over the region

$$x \geq 0, x^2 + y^2 \leq 9.$$



$f_x = 3x^2 - 12 \stackrel{?}{=} 0 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2$

$f_y = 2y \stackrel{?}{=} 0 \Rightarrow y = 0$

$\underbrace{(-2,0)}_{\text{OUTSIDE REGION}}$ or $(2,0)$

BOUNDARIES : $\boxed{A}$ $x = 0 \Rightarrow z = f(0,y) = y^2 \quad -3 \leq y \leq 3$

$2y = 0 \Rightarrow y = 0$ $\quad (0,0)$

$\boxed{B}$ $x^2 + y^2 = 9 \Rightarrow y^2 = 9 - x^2 \Rightarrow z = x^3 - 12x + 9 - x^2 \quad 0 \leq x \leq 3$

$y = \pm\sqrt{9 - x^2}$

$z' = 3x^2 - 12 - 2x \stackrel{?}{=} 0$

$x = \dfrac{2 \pm \sqrt{4 - 4(3)(-12)}}{2(3)} = \begin{matrix} \approx -1.6943 \\ \approx 2.3609 \end{matrix}$

MAX & MIN ~~@~~ OCCUR AT ONE OF THESE

$f(0,3) = 9 \leftarrow$ MAX

$f(0,-3) = 9$

$f(3,0) = 27 - 36 = -9$

$f(2,0) = 8 - 24 = -16 \longleftarrow$ MIN

$f(2.3609, \sqrt{9 - (2.36)^2}) \approx -11.745$

$f(2.3609, -\sqrt{9 - (2.36)^2}) \approx -11.745$

$f(0,0) = 0$

Homework hints

In applied optimization problems,

(a) Identify what you are optimizing!
(b) Label Everything.
(c) Identify given facts (constraints)
(d) Use the constraints and labels to give a 2 variable function for the objective.

HW Examples:

1. Find the points on the cone $z^2 = x^2 + y^2$ that are closest to (4,2,0).

Objective: Minimize **distance** from (x,y,z) points on the cone to the point (4,2,0) given that $z^2 = x^2 + y^2$.

DIST TO (4,2,0) $= \sqrt{(x-4)^2+(y-2)^2+z^2}$

CONSTRAINT: $z^2 = x^2 + y^2$

$\Rightarrow$ $D(x,y) = \sqrt{(x-4)^2+(y-2)^2+x^2+y^2}$

NOW

$$D_x = \frac{1}{2\sqrt{\quad}}\left(2(x-4)+2x\right) \overset{?}{=} 0$$

$$D_y = \frac{1}{2\sqrt{\quad}}\left(2(y-2)+2y\right) \overset{?}{=} 0$$

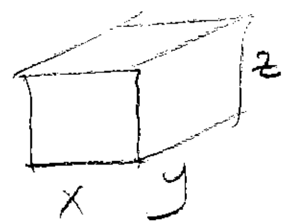
$$4x-8 \overset{?}{=} 0 \Rightarrow x = 2$$

$$4y-4 \overset{?}{=} 0 \Rightarrow y =$$

$\vdots$

2. Find the dimensions of the box with volume 1000 cm³ that has minimum surface area.

Objective: Minimize **surface area** given that volume is 1000.



Surface Area $= 2xy + 2yz + 2xz$

CONSTRAINT: $xyz = 1000 \Rightarrow z = \frac{1000}{xy}$

$S(x,y) = 2xy + 2y \frac{1000}{xy} + 2x \frac{1000}{xy}$

$S(x,y) = 2xy + \frac{2000}{x} + \frac{2000}{y}$

$S_x = 2y - \frac{2000}{x^2} \stackrel{?}{=} 0$

$S_y = 2x - \frac{2000}{y^2} \stackrel{?}{=} 0$

$\Rightarrow y = \frac{1000}{x^2}$

$2x - \frac{2000}{\left(\frac{1000}{x^2}\right)^2} = 0$

$2x\left(\frac{(1000)^2}{x^4}\right) - 2000 = 0$

$\frac{(1000)^2}{x^3} = 1000$

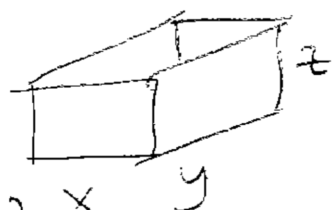
$x^3 = 1000$

$x = (1000)^{1/3}$

⋮

3. You want to build aquariums with slate for the base and glass for the sides (and no top).
Assume slate costs \$5 per in^2 and glass costs \$1 per in^2.
If the volume must be 1000 in^3, then what dimensions will minimize cost?

Objective: Minimize **cost** when volume needs to be 1000.



COST PER in^2 ↓ AREA OF BASE ↓ COST PER in^2 ↓ AREA OF GLASS ↓

$$\text{COST} = 5\,xy + 1\,(2xz + 2yz)$$

$$xyz = 1000 \Rightarrow z = \frac{1000}{xy}$$

$$C(x,y) = 5xy + 2x\frac{1000}{xy} + 2y\frac{1000}{xy}$$

$$C(x,y) = 5xy + \frac{2000}{y} + \frac{2000}{x}$$

$$C_x = 5y - \frac{2000}{x^2} \stackrel{?}{=} 0$$

$$C_y = 5x - \frac{2000}{x^2} \stackrel{?}{=} 0$$